COMPUTING COSET REPRESENTATIVES AND GENERATORS OF $\Gamma_0(3, N)$

ZACHARY PORAT

1. Introduction

Recall that in the case of classical modular forms, for $N \in \mathbf{Z}^+$, we define the **principal congruence subgroup** $\Gamma(2, N)$ to be

$$\Gamma(2, N) = \{\gamma \in \mathrm{SL}(2, \mathbf{Z})\colon \gamma \equiv I \pmod N\}.$$

We call a subgroup $\Gamma \in \mathrm{SL}(2, \mathbf{Z})$ a **congruence subgroup** if Γ contains $\Gamma(N)$ for some N. We call the smallest such N the **level**.

The principal congruence subgroup has finite index in $\mathrm{SL}(2, \mathbf{Z})$. In fact, every congruence subgroup has finite index in $\mathrm{SL}(2, \mathbf{Z})$. Aside from $\Gamma(2, N)$, perhaps the next most important congruence subgroup is the **Hecke congruence subgroup**, denoted $\Gamma_0(2, N)$. This subgroup is defined as

$$\Gamma_0(2, N) = \left\{\gamma \in \mathrm{SL}(2, \mathbf{Z})\colon \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\right\}.$$

[AGG84] constructed an analogous subgroup of $\mathrm{SL}(n, \mathbf{Z})$ with $n \geq 3$. In particular, for the $n = 3$ case, we have

$$\Gamma_0(3, N) = \left\{\gamma \in \mathrm{SL}(3, \mathbf{Z})\colon \gamma \equiv \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \pmod N\right\}.$$

Given the importance of $\Gamma_0(2, N)$, computer algebra systems like SageMath have built-in functionality to compute its generators. However, these programs do not have predefined functions to compute generators of the less-studied $\Gamma_0(3, N)$. In these notes, we describe a SageMath program that uses a finite presentation of $\mathrm{SL}(3, \mathbf{Z})$ and the Reidemeister-Schreier method to produce the generators of $\Gamma_0(3, N)$.

2. The Reidemeister-Schreier Method

Let G be a group and let H be a subgroup of G. The Reidemeister-Schreier method allows us to produce a presentation of H from a presentation of G. In particular, if

G is finitely presented, so too is H. This presentation will provide a (finite) set of generators for H.

For our purposes, we will let $\mathrm{SL}(3,\mathbf{Z})$ be the overlying group, and $\Gamma_0(3,N)$ will be the subgroup of interest. Our goal is to use the Reidemeister-Schreier method to construct a set of generators for $\Gamma_0(3,N)$. Thus, we need to start with a presentation of $\mathrm{SL}(3,\mathbf{Z})$, which can be found in [CRW92].

Theorem 2.1. The group $\mathrm{SL}(3,\mathbf{Z})$ has presentation

$$G = \langle x, y, z \mid x^3 = y^3 = z^2 = (xz)^3 = (yz)^3 = (x^{-1}zxy)^2 = (y^{-1}zyx)^2 = (xy)^6 = 1\rangle.$$

Moreover, G is isomorphic to $\mathrm{SL}(3,\mathbf{Z})$ via the mappings

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$

Now, we will use G to produce a finite presentation of $H \leq G$ where $H \cong \Gamma_0(3,N)$ using the isomorphism above. Let S denote the set of three generators and let $F(S)$ denote the free group on S. We make the following definitions invoking the notation of [Knu17]. Another helpful resource on the subject is [Cas17].

Definition 2.2. A set $A = \{g_\ell\}_{\ell \in F(S)\backslash H}$ of (right) coset representatives is a **Schreier set** for H if, after writing each g_ℓ as a reduced word in S, every initial subword of g_ℓ is again an element of A. Moreover, a **Schreier transversal** is a Schreier set containing exactly one element from each right coset.

Remark. For $g \in F(S)$, let $\overline{g}$ denote the representative of g in $F(S)\backslash H$ as viewed in $F(S)$. That is, the composite map $F(S) \to F(S)\backslash H \cong A \to F(S)$ takes $g \mapsto \overline{g}$.

Theorem 2.3. Let $H \leq F(S)$ be a subgroup and let A a Schreier set for H. Then, the nontrivial elements of the form $g_\ell s(\overline{g_\ell s})^{-1}$ for $\ell \in F(S)\backslash H$ and $s \in S$ freely generate H.

The goal of our code is to build a list of elements of the form $g_\ell s(\overline{g_\ell s})^{-1}$ and then transport these elements back along the isomorphism to produce actual matrix generators for $\Gamma_0(3,N)$. During this process, a Schreier transversal will be constructed and stored. In the end, this program has two possible outputs: one that behaves similarly to the built-in `Gamma0.coset_reps()` command available in SageMath for $\Gamma_0(2,N)$, and another that behaves like the `Gamma0.gens()` command.

3. Explanation of Code

Currently, this code only supports prime level N. To use, load the program by opening SageMath in the same directory as the downloaded file and invoking the command `load("Gamma0_3_Data.py")`. (If the file was renamed after its download, replace `Gamma0_3_Data.py` with the appropriate file name.)

The program contains two commands. The command `Gamma0_3_coset_reps(N)` returns a set of coset representatives for $\Gamma_0(3, N)$ for any prime N input by the user. The command `Gamma0_3_gens(N)` returns a set of generators for $\Gamma_0(3, N)$. If the user chooses an N that is not prime, the program will return an error.

The code is broken down into various blocks. A brief description of what each block does can be found below.

Block 1. In this block, the user sets the level of the modular form N and computes the index of $\Gamma_0(3, N)$ in $\mathrm{SL}(3, \mathbf{Z})$. Additionally, the set of generators S is built and the free group $F(S)$ is constructed. An element $\mathtt{w} \in F(S)$ can be passed along the isomorphism to $\mathrm{SL}(3, \mathbb{Z})$ by using the command `w(M1, M2, M3)`, where

$$\mathtt{M1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathtt{M2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathtt{M3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}.$$

Finally, a list of length one words in $F(S)$ is made, denote this list L_1, and the Schreier set A is initialized with one element, the word x, written in Tietze notation.

Block 2. For each $w \in L_1$, this block checks if w is already represented in A. Let a be a coset representative in A. To check if w is in the same coset as a, we first send w and a through the isomorphism, getting matrices M_w and M_a corresponding to w and a respectively. Then, we examine if the matrix product $M_w M_a^{-1}$ is in $\Gamma_0(3, N)$, i.e. if

$$M_w M_a^{-1} \equiv \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \pmod{N}.$$

If w is in a coset already represented in A, then w is discarded. Otherwise, w represents a new coset and is added to A.

Remark. In future blocks, when we are checking whether a new word is already represented in A, this is the process being performed.

Block 3. The first step in this block is to check whether the size of A is equal to the index of $\Gamma_0(3,N)$. If so, then the construction of A is complete and the program goes to Block 5.

Let L_i denote the list of length i words in A. We start with $i = 1$ from Block 2. For efficiency, we need only build words that start with the length i words already in A; this ensures initial segment closure. We build the potential words of length $i+1$ by taking each word in L_i and concatenating by each generator in S. Let L_{i+1} denote this set of potential words with length $i+1$.

Block 4. For each $w \in L_{i+1}$, we check whether w is in same coset as any representative in A. If so, we discard w. Otherwise, we append w to A. We then return to Block 3, iterating on i.

Recall that in order for a word w to be in a Schreier set A, the word must be closed under initial segments. That is, every initial subword of w must also be in A. Therefore, the maximum length of word possible in A is $[\mathrm{SL}(3,\mathbf{Z})\colon \Gamma_0(3,N)]$. Thus, the number of times the following process of building words can be repeated is capped when the word length i equals this index.

At the completion of this iteration, the resulting set A is a Schreier transveral. This is the point in the code where the command `Gamma0_3_coset_reps(N)` terminates and returns A, which is a set of coset representatives for $\Gamma_0(3,N)$.

Block 5. In this block, each word g_ℓ in A is concatenated by each generator in S on the right. The result is words of the form $g_\ell s$. We then find $\overline{g_\ell s}$, i.e. check which coset in A represents this new word. Next, we compute $(\overline{g_\ell s})^{-1}$ and finally construct a potential generator $g_\ell s(\overline{g_\ell s})^{-1}$. We append these potential generators to a list.

Block 6. The final block converts the potential generators from Tietze notation to elements of $F(S)$, and then passes them along the isomorphism. The identity matrix is also removed in this step. The block finishes by finding the unique matrix generators from the list and formatting them appropriately.

4. Potential Improvements

Mathematically, this process is not very fast in returning a set of generators, nor will it return a minimal generating set. SageMath currently uses Farey symbols to compute a minimal generating set for $\Gamma_0(2,N)$. Using a similar method would allow us to do the same in our case. As for the speed of the program, storing less information or performing smaller calculations would yield faster run times. One way to do this is by computing only the first column of the test matrix, as this is all the data required to check whether a potential element is already represented in A.

References

[AGG84] A. Ash, D. Grayson, and P. Green, *Computations of cuspidal cohomology of congruence subgroups of* $\mathrm{SL}(3,\mathbf{Z})$, J. Number Theory **19** (1984), no. 3, 412–436. (MR769792)

[Cas17] L. Casey, *Reidemeister-Schreier rewriting process for group presentations*, Master's Thesis, Portland State University, 2017. (Link)

[CRW92] M. Conder, E. Robertson, and P. Williams, *Presentations for* 3*-dimensional special linear groups over integer rings*, Proc. Amer. Math. Soc. **115** (1992), no. 1, 19–26. (MR1079696)

[Knu17] B. Knudsen, *Configuration spaces in algebraic topology*: *Lecture 4*, 2017. (Link)